

So, by the Dominated Convergence Theorem,

$$\int_{z \in B(0,1)} \|f(\cdot + rz) - f(\cdot)\|_1 dz \xrightarrow[r \downarrow 0]{} 0,$$

$$\text{so } \|f_r - f\|_1 \xrightarrow[r \downarrow 0]{} 0.$$

The proof is complete.

Hausdorff dimension

A set E may be a subset of \mathbb{R}^n , but saying that it is truly " n -dimensional" may make no actual sense. For example, we feel that a line segment should be a 1-dimensional object, no matter whether we see it as a subset of \mathbb{R} , \mathbb{R}^2 , \mathbb{R}^3 , etc. The sphere $S^2 \subseteq \mathbb{R}^3$ locally looks like \mathbb{R}^2 , so we feel that it should be

2-dimensional object, no matter which larger space it lives in.

The Hausdorff dimension is a notion of dimension that is sensible in the above sense.

In fact, it does not have to be an integer; it can be any non-negative number.

The notion of Hausdorff dimension becomes clearer when we try to understand the Lebesgue measure, and its limitations to provide useful information about some Borel sets.

Def: for any $E \subseteq \mathbb{R}^n$, we define the Lebesgue outer measure of E , $|E|^*$, as

$$|E|^* = \inf \left\{ \sum_{j=1}^{+\infty} \underbrace{|B(x_j, r_j)|}_{c_n r_j^n} : E \subseteq \bigcup_{j=1}^{+\infty} B(x_j, r_j), \text{ where } B(x_j, r_j) \text{ is a ball in } \mathbb{R}^n, \forall j=1, \dots \right\} =$$

$$= \inf \left\{ \sum_{j=1}^{+\infty} c_n r_j^n \right\}, \text{ where the inf is taken over all } \underline{\text{countable}} \text{ covers}$$

of E by balls $B(x_j, r_j)$, $j=1, 2, \dots$ in \mathbb{R}^n =

= $c_n \cdot \inf \left\{ \sum_{j=1}^{+\infty} r_j^n \right\}$, where the inf is taken over all countable covers of E by balls $B(x_j, r_j)$, $j=1, 2, \dots$ in \mathbb{R}^n .

The Lebesgue outer measure is a measure when restricted on the Borel σ -algebra,
i.e. the σ -algebra generated by the open subsets of \mathbb{R}^n .

The power n here, to which the radii of the balls are raised (to give the n -dimensional volume of the corresponding balls) is very important for us now. The reason is that, many times, using the power n there (note that $E \subseteq \mathbb{R}^n$) does not provide any information about the set: more precisely, this happens when $|E|^* = 0$. Let us see an example:

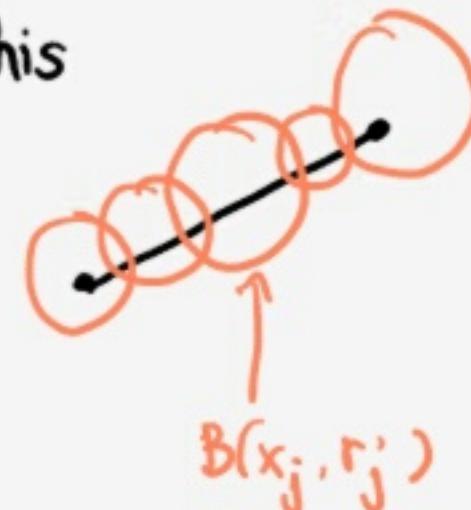
ex: We consider a line segment l in \mathbb{R}^2 :



We know that l is Borel-measurable, and $|l| (= |l|^*) = 0$ (in \mathbb{R}^2 !)

So, $|l| = \underbrace{(\text{constant})}_{\substack{\parallel \\ 0}} \cdot \inf \left\{ \sum_{j=1}^{+\infty} r_j^2 \right\}$, over all countable covers of l
by balls $B(x_j, r_j), j=1, 2, \dots$ in \mathbb{R}^2 .

we don't
care about this
constant



This tells us that l is small in \mathbb{R}^2 ,
but nothing more about how small it is.
Points also have Lebesgue measure 0 in
 \mathbb{R}^2 , but they are much smaller than l .

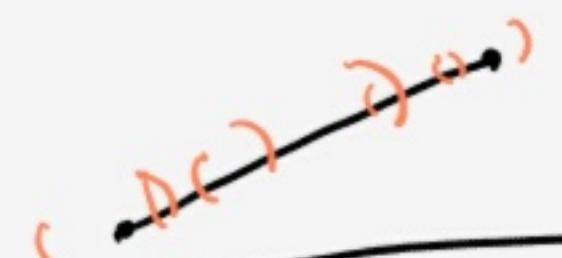
The problem really is that r_j^2 represents (up to multiplication with a constant)
the 2 -dimensional volume of a ball of radius r_j ; while l could have been
approximated by 1 -dimensional balls instead. I.e.:

If we cover l , as before, with 2-dim balls $B(x_j, r_j)$, $j=1, 2, \dots$,

but we look at the inf with powers 1 instead, i.e.

$\inf \left\{ \sum_{j=1}^{+\infty} r_j^1 \right\}$, over all countable covers of l with balls $B(x_j, r_j)$, $j=1, 2, \dots$ in \mathbb{R}^2 ,

then it will be as if we covered l with balls in \mathbb{R}^1 with radii r_j , $j=1, 2, \dots$,



the length of l

and this is really the Lebesgue measure of l when l is seen as a subset of \mathbb{R}^1 !

(up to multiplication with c_1 , but we don't care about that).

So, in this case, using power 1 in the inf makes much more sense than power 2, because

it gives us much more information about the set: mainly, that it can be covered by countably many 1-dim balls. In fact, the Hausdorff dimension of I will turn out to be 1 (not a big surprise!).

And, morally, the Hausdorff dimension of a set will be the power in that inf that will give us the most information about the set: mainly, about the "dimension" of countably many balls required to cover it. This "dimension" can be any non-negative number.

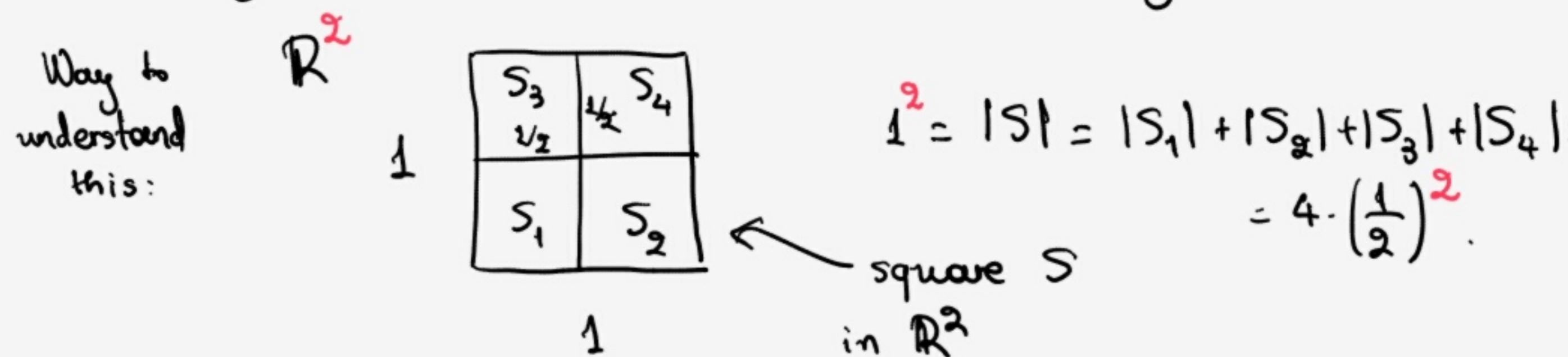


Before we proceed with definitions, we should point out one more thing.

In the definition of the Lebesgue outer measure of $E \subseteq \mathbb{R}^n$

$|E|^* = \inf \left\{ \sum r_j^n \right\}$, where the inf is taken over all countable covers of E by balls $B(x_j, r_j)$, $j=1, 2, \dots$ in \mathbb{R}^n ,

it doesn't matter if we specify that we want all the balls to have radii smaller than some number $\varepsilon > 0$ or not. The reason is that the volume of any ball can be approximated by the volumes of smaller balls covering it.



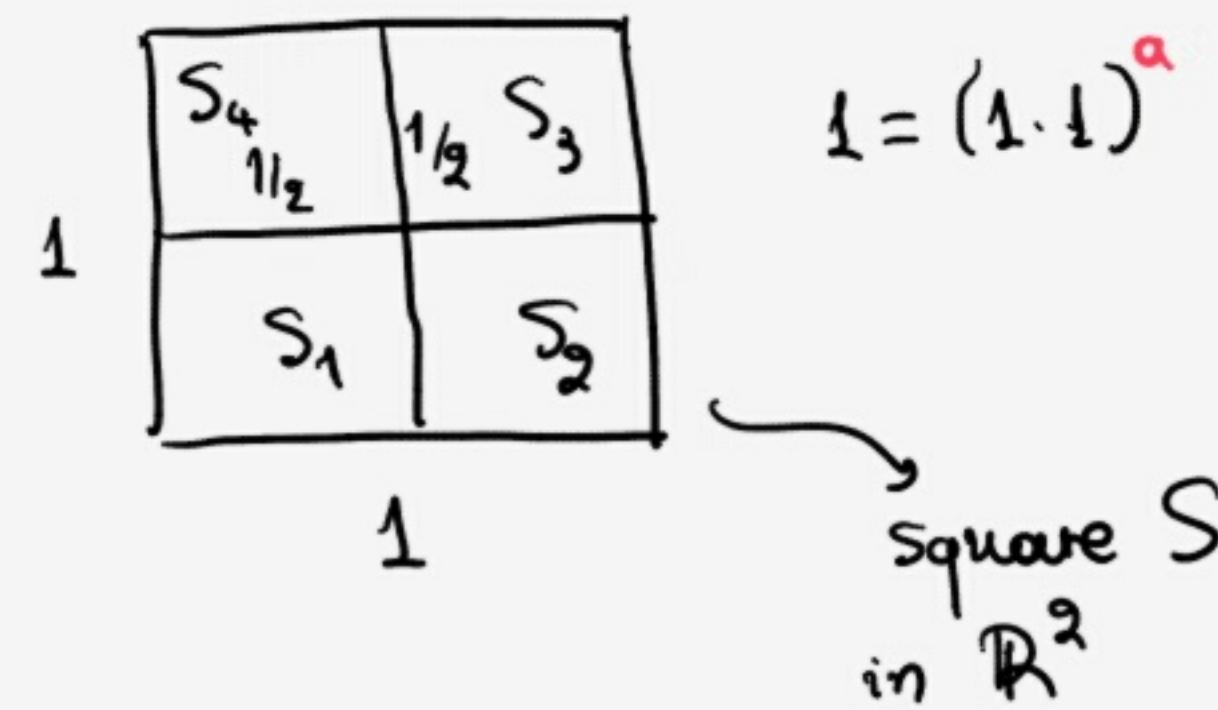
$$1^2 = |S| = |S_1| + |S_2| + |S_3| + |S_4| = 4 \cdot \left(\frac{1}{2}\right)^2.$$

So, even if we take the inf over fewer covers, where the balls have radii smaller than some $\varepsilon > 0$, the inf stays the same.

However, this is not true when we use a different power in the inf! In general, the inf $\inf \left\{ \sum_{j=1}^{+\infty} r_j^\alpha \right\}$, over all countable covers of E by balls $B(x_j, r_j)$, $j=1, 2, \dots$ in R^n , where $r_j < \varepsilon \quad \forall j=1, 2, \dots$,

goes larger and larger as $\varepsilon \searrow 0$ (the constraint that the balls should be small means that we are getting fewer covers, so the inf is taken over a smaller set).

Indication of this: $\mathbb{R}^2, \alpha < 2$:



$1 = (1 \cdot 1)^\alpha$, but $4 \cdot \left(\frac{1}{2}\right)^\alpha \gg 1$, so the "size" of the large set S cannot be approximated by the "sizes" of smaller sets covering it.

At the same time, we care to understand sets locally, so it makes sense to consider ε small.

After this discussion, the following definition will hopefully feel natural.

Def: Let $\alpha > 0$, $\epsilon > 0$. For all $E \subseteq \mathbb{R}^n$, we define

the power in the inf

$$H_E^\alpha := \inf \left\{ \sum_{j=1}^{+\infty} r_j^\alpha \right\},$$

where the inf is taken over all countable covers of E by balls $B(x_j, r_j)$ in \mathbb{R}^n , $j=1, 2, \dots$, with $r_j < \epsilon$ $\forall j=1, 2, \dots$

↓
the upper
bound on the
radii of balls
in allowed covers.

[As we have already mentioned, the Hausdorff dimension of E will be the α
that gives the most information about E as ϵ grows smaller and smaller.]

Properties of $H_\varepsilon^\alpha(E)$: By the earlier discussion, the following hold:

- If $E \subseteq \mathbb{R}^n$, $H_\varepsilon^n(E) = \frac{1}{c_n} \cdot |E|^*$, $\forall \varepsilon > 0$

(and $H_\varepsilon^n(E) = \frac{1}{c_n} |E|$ if E is Borel-measurable).

- If $\alpha > 0$, $\forall \varepsilon > 0$, $\forall E \subseteq \mathbb{R}^n$,
 $H_\varepsilon^\alpha(E)$ increases as $\varepsilon \downarrow 0$.
→ not necessarily strictly increasing (ex.: $\alpha = n$)

(the reason is that, as $\varepsilon \downarrow 0$, we allow fewer covers, so we consider the inf of a smaller set).

Lecture 17 (26/11/2014)

for all $\alpha > 0$, $\varepsilon > 0$, $E \subseteq \mathbb{R}^n$, we have defined

$H_\alpha^\varepsilon(E) := \inf \left\{ \sum_{j=1}^{+\infty} r_j^\alpha \right\}$, over all countable covers of E by balls $B(x_j, r_j)$, $j=1, 2, \dots$ in \mathbb{R}^n ,
with $r_j < \varepsilon$ $\forall j=1, 2, \dots$.

Properties of H_α^ε (for fixed n):

- $H_n^\varepsilon(E) = c_n \cdot |E|^*$, $\forall E \subseteq \mathbb{R}^n$, where $|E|*$ is the Lebesgue outer measure of E as a subset of \mathbb{R}^n , and c_n is a constant depending only on n .
- fix $\alpha > 0$, $E \subseteq \mathbb{R}^n$. $H_\alpha^\varepsilon(E)$ increases as ε decreases.
- Fix $\alpha > 0$, $\varepsilon > 0$. H_α^ε is an **outer measure** on the subsets of \mathbb{R}^n :

- $H_\alpha^\varepsilon(\emptyset) = 0$, as \emptyset can be covered by any ball of arbitrarily small radius.
- $\forall A \subseteq B$ in \mathbb{R}^n , $H_\alpha^\varepsilon(A) \leq H_\alpha^\varepsilon(B)$, because any cover of B is a cover of A , and thus $H_\alpha^\varepsilon(A)$ is the infimum of a larger set.

- If $E \subseteq \bigcup_{i=1}^{+\infty} E_i$ in \mathbb{R}^n , then $H_\alpha^\varepsilon(E) \leq \sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i)$. Indeed, let $\delta > 0$, and,

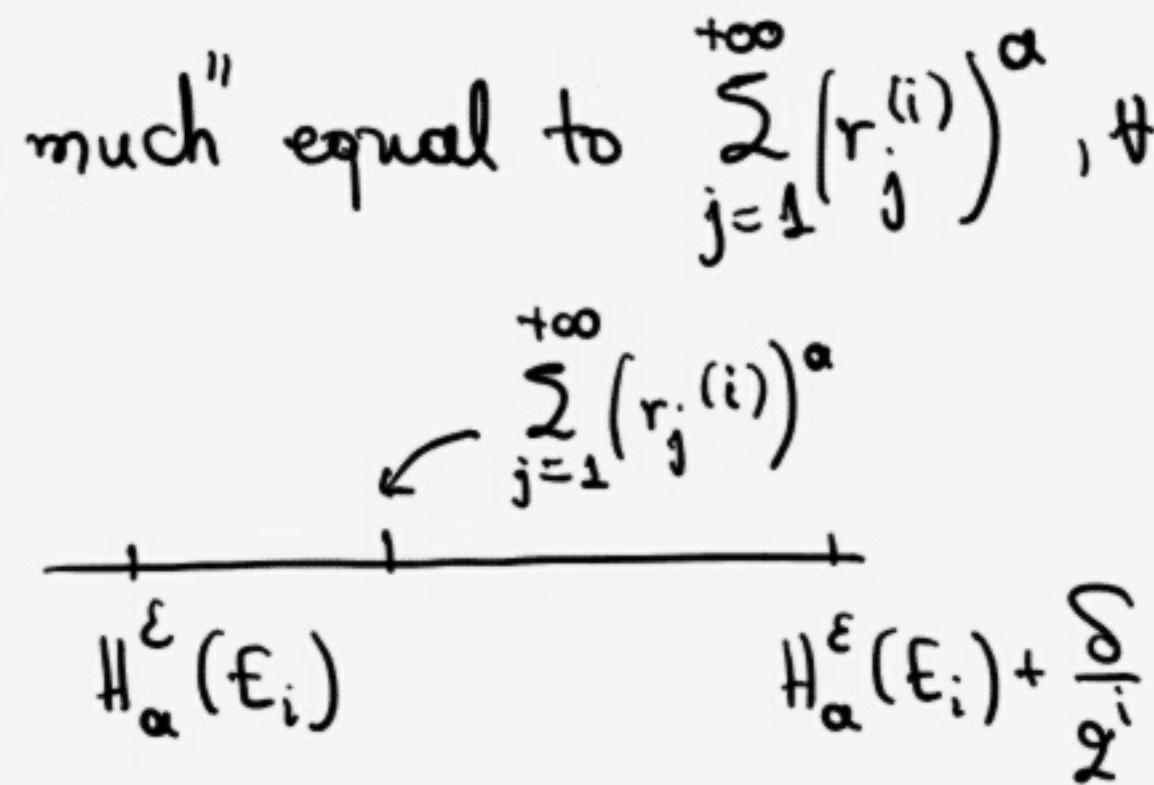
for each $i = 1, 2, \dots$, B_i be a countable cover of E_i by balls $B(x_j^{(i)}, r_j^{(i)})$ in \mathbb{R}^n ,

with $r_j^{(i)} < \varepsilon \forall j = 1, 2, \dots$, s.t. $H_\alpha^\varepsilon(E_i)$ is "pretty much" equal to $\sum_{j=1}^{+\infty} (r_j^{(i)})^\alpha$, $\forall i$,

i.e. $\forall i$:

$$\sum_{j=1}^{+\infty} (r_j^{(i)})^\alpha < H_\alpha^\varepsilon(E_i) + \frac{\delta}{2^i}$$

(such B_i exist by the properties of an infimum).



Now, the union $\bigcup_{i=1}^{+\infty} B_i$ is a cover for E , so

$$\begin{aligned} H_\alpha^\varepsilon(E) &\leq \sum_{\substack{\text{balls in} \\ \bigcup_{i=1}^{+\infty} B_i}} (\text{radii of these balls})^\alpha = \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (r_j^{(i)})^\alpha \leq \sum_{i=1}^{+\infty} \left(H_\alpha^\varepsilon(E_i) + \frac{\delta}{2^i} \right) \\ &\leq \left(\sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i) \right) + \delta. \end{aligned}$$

And since δ was arbitrary, we have $H_\alpha^\varepsilon(E) \leq \sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i)$.

- Let $a > b$, $\varepsilon > 0$. Then, $H_a^\varepsilon(E) \leq \varepsilon^{a-b} H_b^\varepsilon(E)$, $\forall E \subseteq \mathbb{R}^n$.

In particular, for $\varepsilon < 1$, $\varepsilon^{a-b} < 1$, so $H_a^\varepsilon(E) \leq H_b^\varepsilon(E)$, $\forall E \subseteq \mathbb{R}^n$.

Proof: Let $E \subseteq \mathbb{R}^n$. Suppose that the balls $B(x_j, r_j)$, $j=1, 2, \dots$ cover E , with $r_j < \varepsilon$

$$\forall j=1, 2, \dots. \text{ Then: } \sum_{j=1}^{+\infty} r_j^a = \sum_{j=1}^{+\infty} r_j^{a-b} r_j^b \stackrel{r_j < \varepsilon}{\leq} \varepsilon^{a-b} \cdot \sum_{j=1}^{+\infty} r_j^b, \text{ so } H_a^\varepsilon(E) \leq \varepsilon^{a-b} H_b^\varepsilon(E).$$

We are interested in understanding sets $E \subseteq \mathbb{R}^n$ locally, and thus in $H_\alpha^\varepsilon(E)$ for ε very small.
 So, we need to understand the following:

Def.: Let $\alpha > 0$, $E \subseteq \mathbb{R}^n$. We define

$$H_\alpha(E) := \lim_{\varepsilon \rightarrow 0} H_\alpha^\varepsilon(E) \quad \left(= \sup_{\varepsilon > 0} H_\alpha^\varepsilon(E) \right).$$

Properties of $H_\alpha(E)$ (for fixed n):

- $H_n(E) = c_n |E|^*$ ($= H_n^\varepsilon(E) \forall \varepsilon > 0$),
 where $|E|*$ is the Lebesgue outer measure of E in \mathbb{R}^n , and c_n is a constant depending only on n .
- fix $\alpha > 0$. H_α is an outer measure on the subsets of \mathbb{R}^n :
 - $H_\alpha(\emptyset) = 0$, as $H_\alpha(\emptyset) = \lim_{\varepsilon \rightarrow 0} \underbrace{H_\alpha^\varepsilon(\emptyset)}_0 = 0$.

- If $A \subseteq B$, $H_\alpha(A) \leq H_\alpha(B)$: $H_\alpha^\varepsilon(A) \leq H_\alpha^\varepsilon(B)$ $\forall \varepsilon > 0$ (as H_α^ε is an outer measure),

$$\text{so } \lim_{\varepsilon \rightarrow 0} H_\alpha^\varepsilon(A) \leq \lim_{\varepsilon \rightarrow 0} H_\alpha^\varepsilon(B).$$

$$\begin{array}{ccc} || & & || \\ H_\alpha(A) & & H_\alpha(B) \end{array}$$

- If $E \subseteq \bigcup_{i=1}^{+\infty} E_i$, then $H_\alpha(E) \leq \sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i)$:

$$H_\alpha^\varepsilon(E) \leq \sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i) \quad \forall \varepsilon > 0 \quad (\text{as } H_\alpha^\varepsilon \text{ is an outer measure}),$$

$$\text{so } \sup_{\varepsilon > 0} H_\alpha^\varepsilon(E) \leq \sup_{\varepsilon > 0} \sum_{i=1}^{+\infty} H_\alpha^\varepsilon(E_i) \leq \sum_{i=1}^{+\infty} \sup_{\varepsilon > 0} H_\alpha^\varepsilon(E_i).$$

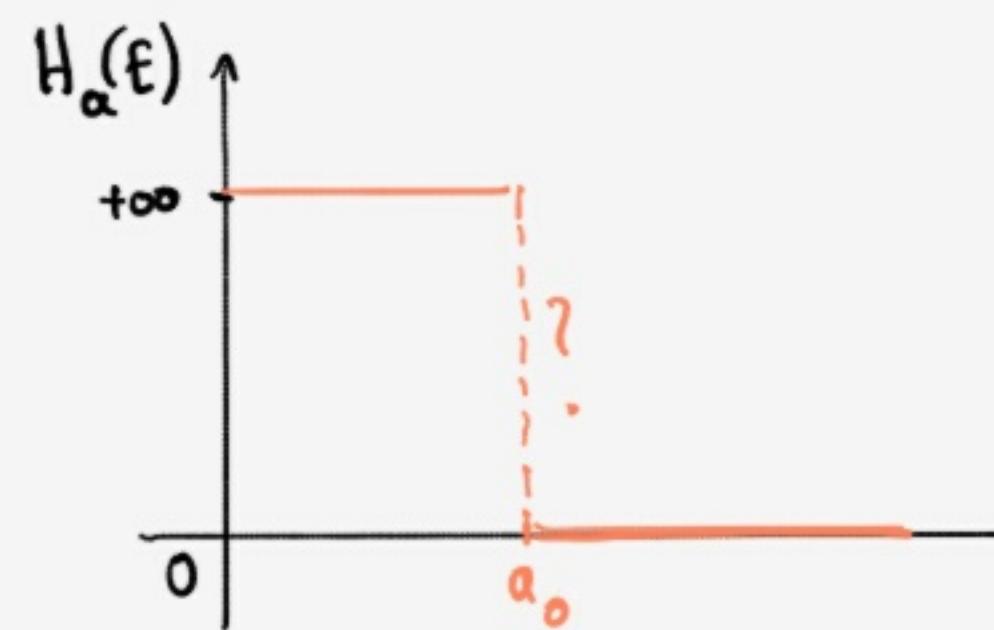
$$\begin{array}{ccc} || & & || \\ H_\alpha(E) & & H_\alpha(E_i) \end{array}$$

note that explaining
this for \lim instead
of \sup requires the Monotone
Convergence Theorem

- fix $\alpha > 0$. H_α is a measure on the Borel σ -algebra of \mathbb{R}^n .
 (We will not prove this, as it requires knowledge on the Caratheodory construction of measures).

Our aim: To understand the behaviour of $H_\alpha(E)$ as α changes. In particular, we will show the following:

$E \subseteq \mathbb{R}^n$, n fixed



if $0 < \alpha_0 \leq n$, s.t.

$$H_\alpha(E) = \begin{cases} +\infty & \text{for } 0 < \alpha < \alpha_0, \\ 0 & \text{for } \alpha > \alpha_0, \\ \text{anything in } [0, +\infty] & \text{for } \alpha = \alpha_0. \end{cases}$$

Remember that α in $H_\alpha(E) = \lim_{\varepsilon \rightarrow 0} H_\alpha^\varepsilon(E)$ is the power the radii are raised to, in the definition of $H_\alpha^\varepsilon(E)$. The picture above implies that something very interesting happens for $\alpha = \alpha_0$: in fact, that α_0 is the power that gives the most information about E , amongst all other α 's. We will define α_0 to be the **Hausdorff dimension** of E .

We now start revealing the picture above.

- $H_\alpha(\mathbb{R}^n) = 0$, $\forall \alpha > n$ (note that $H_n(\mathbb{R}^n) = c_n \cdot \underbrace{|\mathbb{R}^n|}_{\downarrow} = +\infty$)

\downarrow
the Lebesgue measure
of \mathbb{R}^n as a subset of \mathbb{R}^n

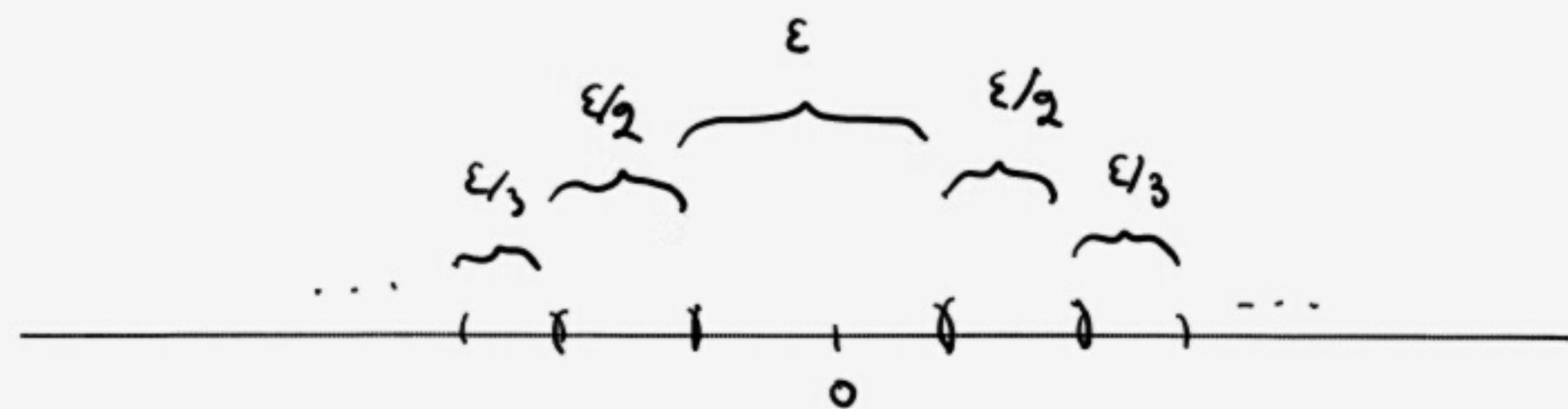
Proof: We prove this for $n=1$ (Attention: Here we mean that we have fixed the ambient space to be \mathbb{R}^1 , and that

$H_\alpha^\varepsilon(E)$ is defined for all $E \subseteq \mathbb{R}^1$,
by taking covers of E by balls in
 \mathbb{R}^1 , i.e. by intervals).

Indeed, let $\alpha > 1 (= n)$. We need to understand $H_\alpha^\varepsilon(\mathbb{R})$, $\varepsilon > 0$.

Let $\varepsilon > 0$. Since $n=1$, we consider covers of \mathbb{R} by balls in \mathbb{R}^1 , i.e. by intervals.

Indeed, we cover \mathbb{R} like this:



This actually covers \mathbb{R} , because $\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{4} + \dots = \varepsilon \cdot \underbrace{\left(\sum_{j=1}^{+\infty} \frac{1}{j} \right)}_{+\infty} = +\infty$.

Now, let \mathcal{B} be this cover. We have that

$$\begin{aligned}
 H_\alpha^\varepsilon(\mathbb{R}) &\leq \sum_{\substack{\text{interval} \\ \text{in } \mathcal{B}}} (\text{radius of interval})^\alpha < \sum_{\substack{\text{interval} \\ \text{in } \mathcal{B}}} (\text{length of interval})^\alpha = \\
 &= \varepsilon^\alpha + 2 \cdot \left(\frac{\varepsilon}{2}\right)^\alpha + 2 \cdot \left(\frac{\varepsilon}{3}\right)^\alpha + 2 \cdot \left(\frac{\varepsilon}{4}\right)^\alpha + \dots = \\
 &= \varepsilon^\alpha \cdot \left(1 + 2 \cdot \frac{1}{2^\alpha} + 2 \cdot \frac{1}{3^\alpha} + 2 \cdot \frac{1}{4^\alpha} + \dots \right) < \\
 &< 2\varepsilon^\alpha \cdot \left(1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} + \dots \right) = 2 \cdot \varepsilon^\alpha \cdot \underbrace{\sum_{j=1}^{+\infty} \frac{1}{j^\alpha}}_{<+\infty, \text{ as } \alpha > 1} \xrightarrow[\varepsilon \rightarrow 0]{} 0, \quad \text{so}
 \end{aligned}$$

$$H_\alpha(\mathbb{R}) = \lim_{\varepsilon \rightarrow 0} H_\alpha^\varepsilon(\mathbb{R}) = 0, \quad \forall \alpha > 1.$$

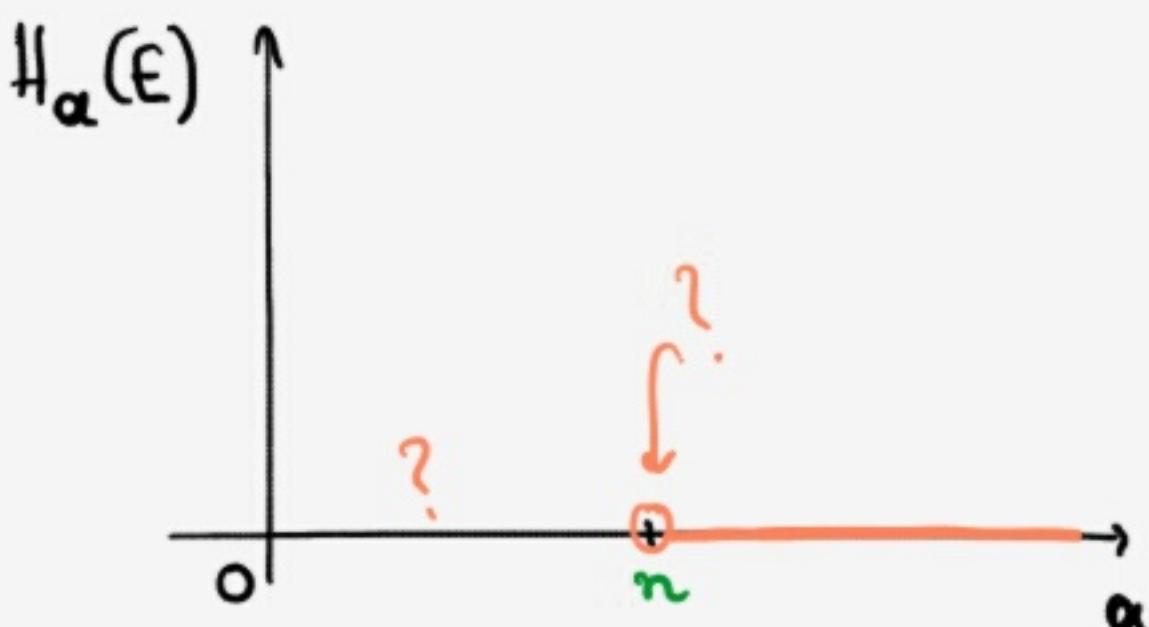
- $\forall E \subseteq \mathbb{R}^n, H_\alpha(E) = 0 \iff \alpha > n$.

This follows from the fact that $E \subseteq \mathbb{R}^n \implies H_\alpha(E) \leq H_\alpha(\mathbb{R}^n) \quad \forall \alpha > 0$.

$$\begin{aligned} &\sim\sim \\ &\parallel \\ &0 \text{ for } \alpha > n. \end{aligned}$$

By the above, so far we have the following picture:

Here, E is seen as
a subset of \mathbb{R}^n , i.e.
 n is the dimension
of the balls used to
cover E in the definition
of $H_\alpha^\varepsilon(E) \quad \forall \varepsilon > 0$.



$$H_\alpha(E) = 0 \quad \text{if } \alpha \geq n.$$

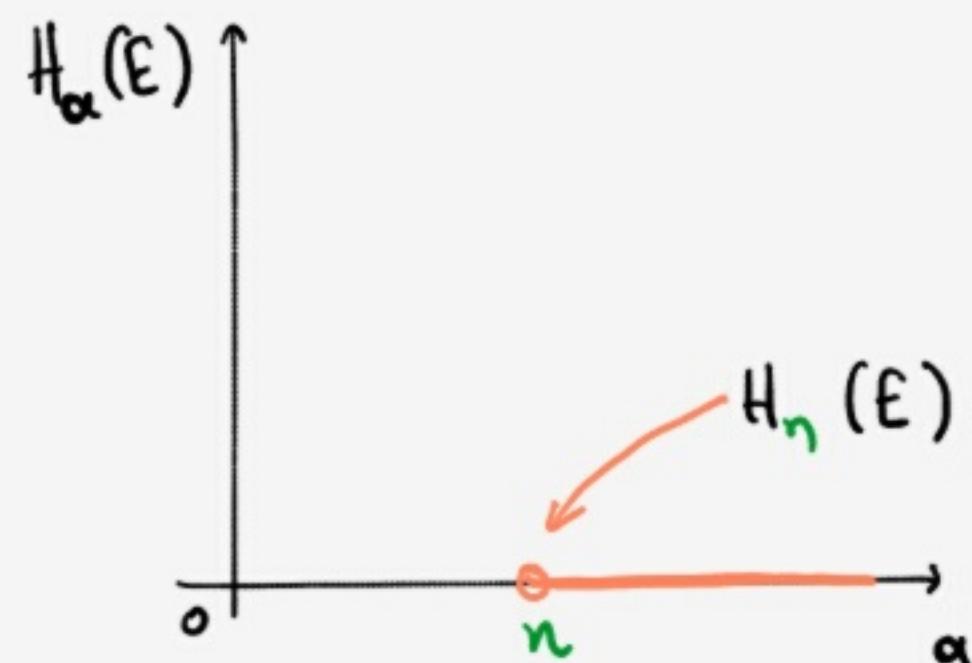
$H_n(E)$ is a multiple of $|E|^n$, so
it may be any value in $[0, +\infty]$.

↳ This picture means that the α , we were talking about earlier, where the "jump" of $H_\alpha(E)$ from $+\infty$ to 0 happens, is $\leq n$. So, the Hausdorff dimension of $E \subseteq \mathbb{R}^n$ is at most n : so this notion of dimension makes sense.

Lecture 18 (27/11/2014)

We have so far shown the following picture :

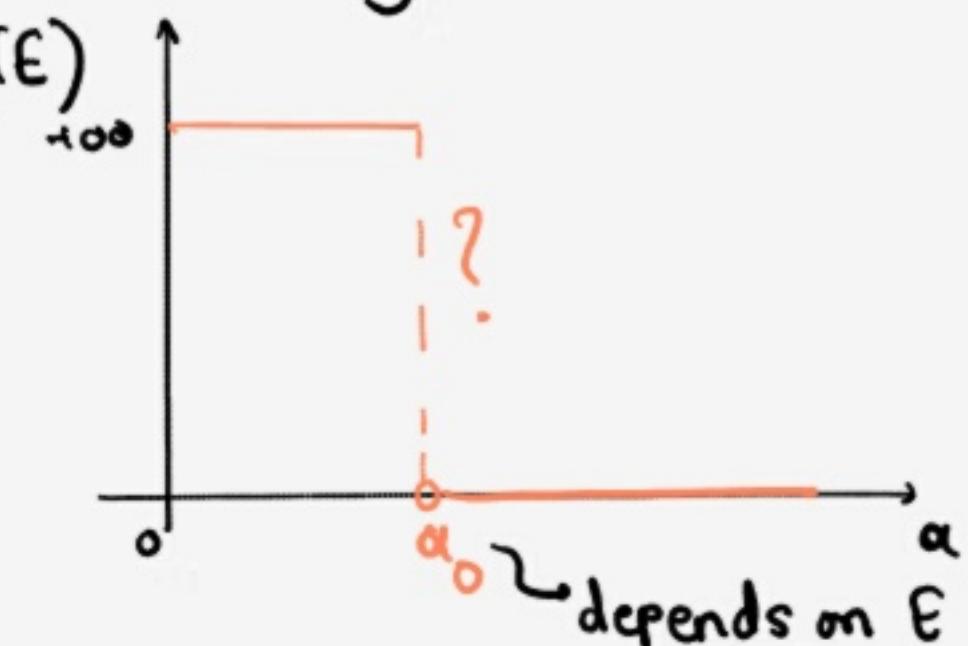
for $E \subseteq \mathbb{R}^n$:



$H_n(E) = c_n \cdot |E|^*$, for some constant c_n depending only on n , where $|E|^\ast$ is the Lebesgue outer measure of E as a subset of \mathbb{R}^n .

Our aim: To show that the following holds, for some α_0 :

For $E \subseteq \mathbb{R}^n$:



$$H_\alpha(E) = \begin{cases} +\infty, & \text{for } \alpha < \alpha_0 \\ 0, & \text{for } \alpha > \alpha_0 \\ \text{an element of } [0, +\infty], & \text{for } \alpha = \alpha_0 \end{cases}$$

(By the picture above, $n \in [\alpha_0, +\infty)$)

It will suffice to prove the following:

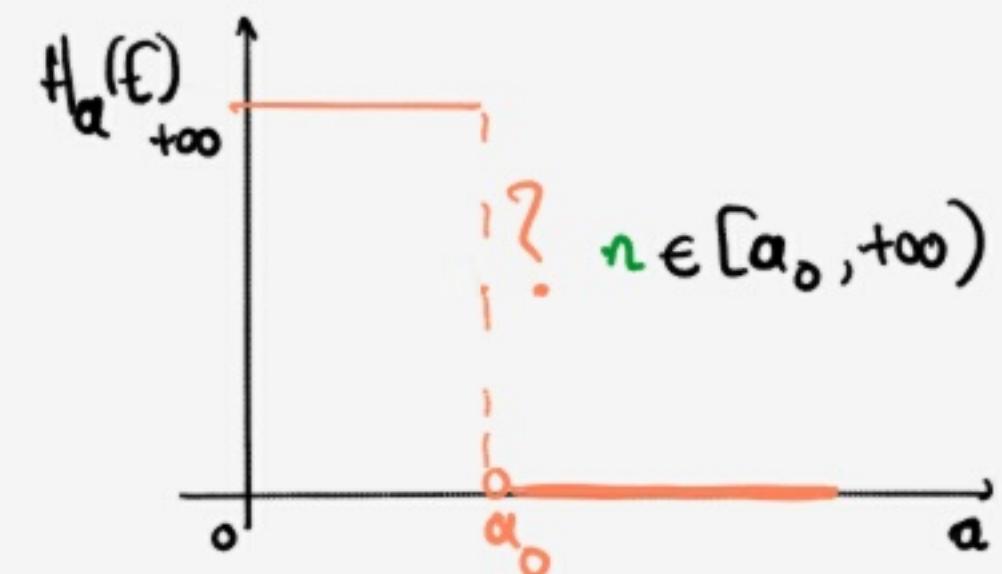
- Let $E \subseteq \mathbb{R}^n$. If $b > 0$ is s.t. $H_b(E) < +\infty$, then $H_a(E) = 0 \quad \forall a > b$.

Proof: We have explained that, $\forall \varepsilon > 0$ and $a > b$:

$$\begin{array}{ccc} H_a^\varepsilon(E) & \leq & \underbrace{\varepsilon^{\alpha-b}}_{\substack{\downarrow \varepsilon \rightarrow 0 \\ 0}} \cdot \underbrace{H_b^\varepsilon(E)}_{\substack{\downarrow \varepsilon \rightarrow 0 \\ 0}} & \xrightarrow[\varepsilon \rightarrow 0]{} 0 \\ H_a(E) & = & 0 & H_b(E), \boxed{< +\infty} \end{array}$$

So, $H_a(E) = 0 \quad \forall a > b$.

What does this mean? It means that, as α increases, once $H_\alpha(E)$ gets a finite value for $\alpha = \text{some } a_0$ (something which holds for $\alpha > n$ but may even happen for $\alpha \leq n$), then $H_\alpha(E) = 0$ for all $\alpha > a_0$. So, we have reached our aim:



The α_0 where the "jump" of $H_\alpha(E)$ from $+\infty$ to 0 happens will be defined as the Hausdorff dimension of E . More precisely:

Def: Let $E \subseteq \mathbb{R}^n$. We define the Hausdorff dimension of E , $\dim_H E$, as

$$\dim_H E := \sup \{\alpha > 0 : H_\alpha(E) = +\infty\} \quad (= \inf \{\alpha > 0 : H_\alpha(E) = 0\}).$$



We have in the definition above that $E \subseteq \mathbb{R}^n$. But perhaps it is not so clear how this n plays a role in the definition. Well, it does: when calculating each $H_\alpha^\varepsilon(E)$ (to in turn find $H_\alpha(E)$), we cover E by balls in \mathbb{R}^n . (note that covering E by countably many balls can certainly be done when the balls we use are n -dimensional, but is not necessarily plausible when we use lower dimensional balls: for example, a disc cannot be covered by

countably many intervals). So:

What is the connection between n and $\dim_H E$?

$H_\alpha(E) = 0 \quad \forall \alpha > n$, so $n \geq \dim_H E$.

If $n < \dim_H E$, then, for

$\alpha \in (n, \dim_H E)$, $H_\alpha(E) = 0$, contradiction,
as $\alpha < \dim_H E \Rightarrow H_\alpha(E) = +\infty$.

In more detail:

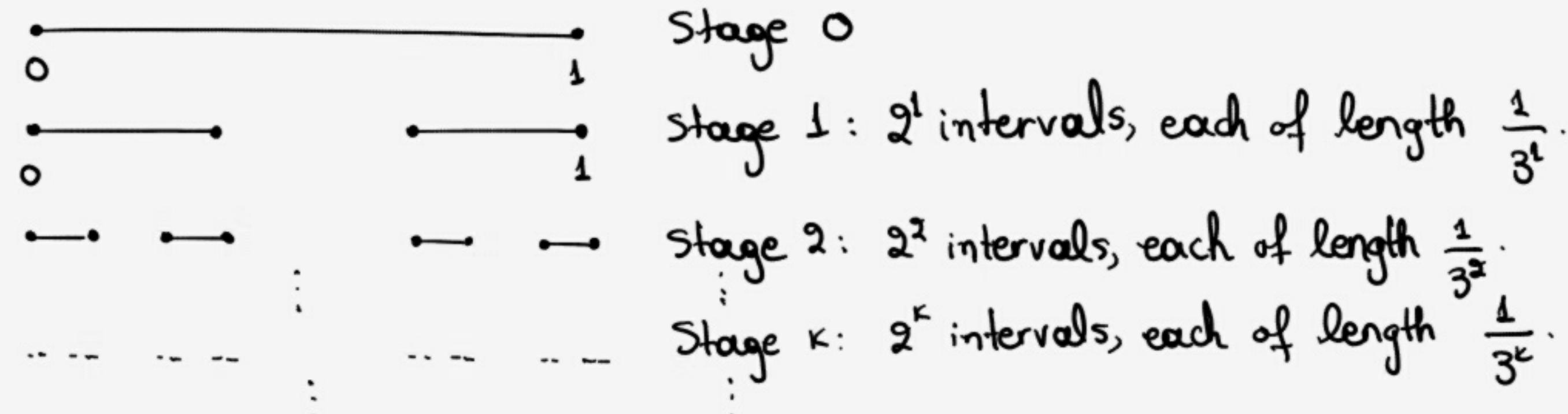
- If $\underbrace{|E|^*}_{\downarrow} \geq 0$, then $\dim_H E = n$: We already know that $\dim_H(E) \leq n$. And if $\dim_H E \leq n$, then $H_n(E) = 0 \Rightarrow |E|^* = 0$,
 \uparrow
 $c_n |E|^*$ contradiction.

- If $|E|^t = 0$, then the situation becomes much more interesting: $\dim_H E$ can be any number in $[0, \infty]$.

Exercise: The middle thirds Cantor set has Hausdorff dimension $\frac{\log 2}{\log 3}$.

Solution: The middle thirds Cantor set C is constructed as follows:

We start with the interval $[0, 1]$. At each stage of the construction, we delete the open middle third of each interval we have:



for each k , C is contained in the intervals of Stage k ,

i.e. in 2^k intervals, each of length $\frac{1}{3^k}$.

• $\dim_H C \leq \frac{\log 2}{\log 3}$: We will show that $H_{\frac{\log 2}{\log 3}}(C) < \infty$.

$C \subseteq \mathbb{R}^1$, so to calculate $H_{\frac{\log 2}{\log 3}}(C)$ we look at $H_{\frac{\log 2}{\log 3}}^{\frac{1}{3^k}}(C)$ as $k \rightarrow \infty$, by covering C by intervals.

Since, $\forall k \in \mathbb{N}$, C is contained in 2^k intervals, each of length $\frac{1}{3^k}$, i.e. radius $\frac{1}{2} \cdot \frac{1}{3^k}$,

we have that $H_{\frac{\log 2}{\log 3}}^{\frac{1}{3^k}}(C) \leq 2^k \cdot \left(\frac{1}{2} \cdot \frac{1}{3^k}\right)^{\frac{\log 2}{\log 3}} < 2^k \cdot \left(\frac{1}{3^k}\right)^{\frac{\log 2}{\log 3}} = 2^k \cdot \left(\frac{1}{3^{\frac{\log 2}{\log 3}}}\right)^k = 2^k \cdot \left(\frac{1}{2}\right)^k = 1$,

so $H_{\frac{\log 2}{\log 3}}(C) \leq 1 < \infty$. So, $\dim_H C \leq \frac{\log 2}{\log 3}$: if $\dim_H C$ was $\geq \frac{\log 2}{\log 3}$, we would

hence that $H_{\frac{\log 2}{\log 3}}(C) = \infty$, a contradiction.

- $\dim_H C \geq \frac{\log 2}{\log 3}$: We will show that $H_{\frac{\log 2}{\log 3}}(C) \geq \frac{1}{2}$,

by showing that $H_{\frac{\log 2}{\log 3}}^\varepsilon(C) \geq \frac{1}{9} - \varepsilon$, $\forall \varepsilon > 0$.

Indeed, let $\varepsilon > 0$. We pick a cover B^* of C by countably many balls in \mathbb{R}^1 (i.e. intervals), each of length $< \varepsilon$, s.t. $\sum_{\substack{\text{balls} \\ \text{in } B^*}} (\text{radius of ball})^{\frac{\log 2}{\log 3}}$ is ε -close to $H_{\frac{\log 2}{\log 3}}^\varepsilon(C)$

(there exists such a cover by the properties of an infimum),

i.e.

$$H_{\frac{\log 2}{\log 3}}^\varepsilon(C) \geq \sum_{\substack{\text{balls} \\ \text{in } B^*}} (\text{radius of ball})^{\frac{\log 2}{\log 3}} - \varepsilon.$$

Now, C is compact, so there exists a finite subcover B of B^* .

$$\text{Clearly, } H_{\frac{\log 2}{\log 3}}^\varepsilon(C) \geq \sum_{\substack{\text{balls} \\ \text{in } B}} (\text{radius of ball})^{\frac{\log 2}{\log 3}} - \varepsilon \geq$$

$$\geq \sum_l \underbrace{\#\{ \text{balls in } B \text{ with radius in } \left[\frac{1}{3^l}, \frac{1}{3^{l-1}} \right] \}}_{B_l} \cdot \underbrace{\left(\frac{1}{3^l} \right)^{\frac{\log 2}{\log 3}}}_{\frac{1}{2^l}} - \varepsilon =$$

$$= \sum_l \# B_l \cdot \frac{1}{2^l} - \varepsilon.$$

this is the sum over all l s.t. $\# B_l \neq 0$, i.e. $B_l \neq \emptyset$. So, it is a finite sum.

We pick k s.t. $\frac{1}{3^k} < \min \left\{ \frac{1}{3^l} : l \text{ s.t. } B_l \neq \emptyset \right\}$. This way, the k -th stage

of the construction of C comes after the l -th stage, for all l s.t. $B_l \neq \emptyset$, i.e.
for all l participating in the sum above.

We will show that $\sum_l \#B_l \cdot \frac{1}{2^l} \geq \frac{1}{2}$,

by showing that $\sum_l \#B_l \cdot 2 \cdot \underbrace{2^{k-l}}_{\text{the number of intervals at the } k\text{-th stage.}} \geq \underbrace{2^k}_{\text{the number of intervals on an interval at the } l\text{'th stage}}$.

note that here we add over l s.t. $B_l \neq \emptyset$, which has a chance of happening only for $l \leq k$.

the number of intervals on an interval at the l 'th stage has split into in the k -th stage, for $l \leq k$.

the number of intervals at the k -th stage.

The idea is that each of the 2^k intervals at the k -th stage is intersected by at least one ball in $B = \bigcup_{l \leq k} B_l$, and each ball in each B_l intersects at most 2 intervals in

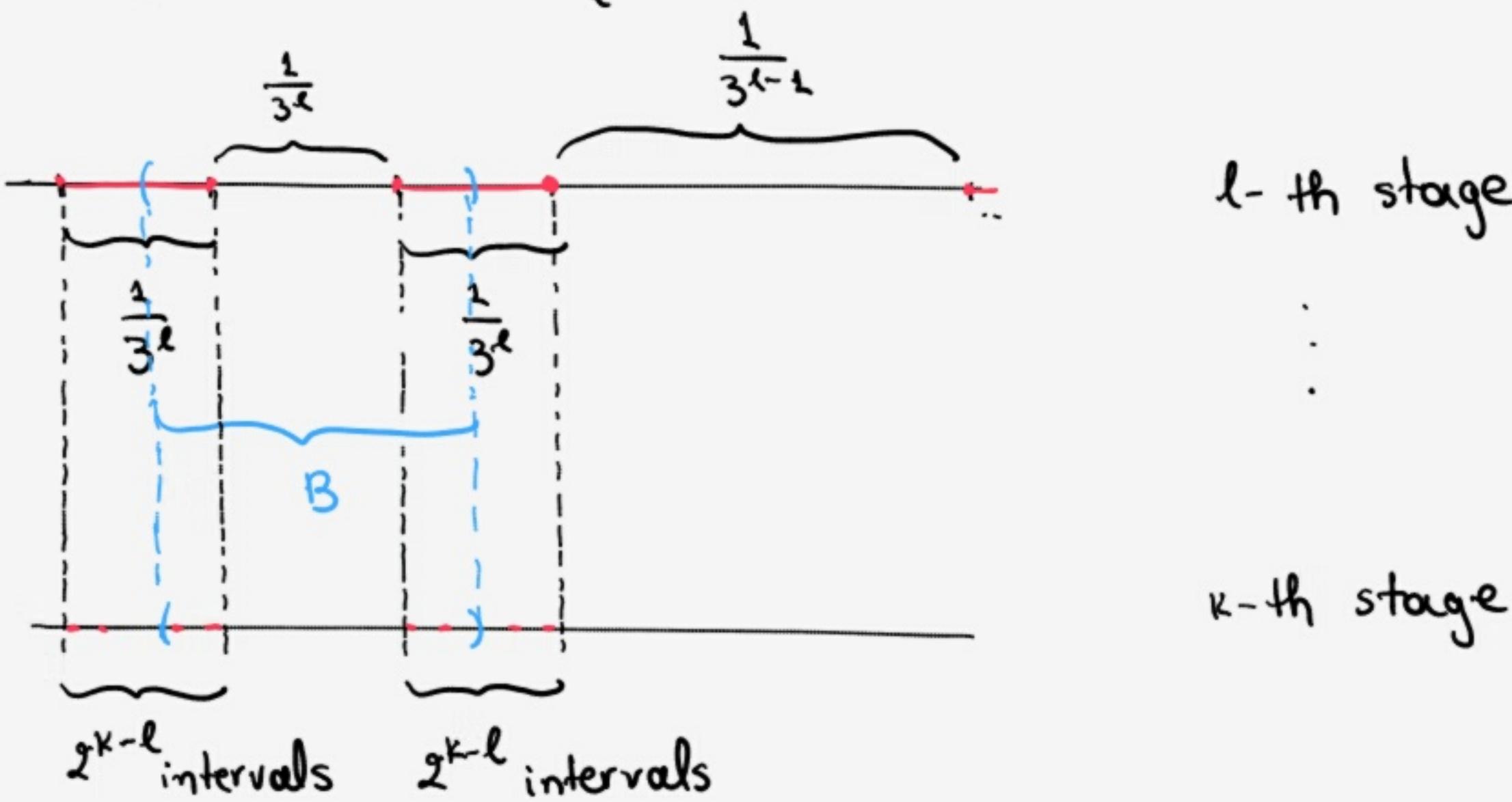
the l -th stage, and thus at most $2 \cdot 2^{k-l}$ intervals at the k -th stage. Indeed:

- Each interval in the k -th stage contains an element of C , and C is covered by the balls (intervals) in B . So, each interval at the k -th stage is intersected by a ball in B .

$$\begin{aligned} \text{So, } 2^k &= \sum_{\substack{\text{intervals} \\ \text{at } k\text{-th} \\ \text{stage}}} \sum_{\substack{\text{intervals} \\ \text{at } k\text{-th} \\ \text{stage}}} \#\{\text{balls in } B \text{ intersecting the interval}\} = \\ &= \sum_{\substack{\text{balls} \\ \text{in } B}} \#\{\text{intervals at } k\text{-th stage that the ball intersects}\} = \\ &\leq \sum_{l < k} \#B_l \cdot \#\{\text{intervals at } k\text{-th stage that a ball in } B_l \text{ can intersect}\}. \end{aligned}$$

Now, each ball in B_l can intersect $\leq 2 \cdot 2^{k-l}$ intervals of the k -th stage.

Indeed, let B be a ball (interval) in B_ℓ .



B has length $\frac{1}{3^{k-1}}$: now, any 3 consecutive intervals at the l -th stage have the property that there exist 2 consecutive ones of these intervals with a gap of length $\geq \frac{1}{3^{l-1}}$ between them.

So, B can intersect ≤ 2 of the intervals of the l -th stage, each of which contains

2^{k-l} intervals of the k -th stage.

So, B can intersect $\leq 2 \cdot 2^{k-l}$ intervals of the k -th stage.

$$\text{So, } 2^k \leq \sum_{l < k} \#B_l \cdot 2 \cdot 2^{k-l} \rightarrow$$

$$\rightarrow \sum_{l < k} \#B_l \cdot \frac{1}{2^l} \geq \frac{1}{2}.$$

We have proven that $H_{\frac{\log 2}{\log 3}}^\varepsilon(C) \geq \frac{1}{2} - \varepsilon \quad \forall \varepsilon > 0,$

$$\text{so } H_{\frac{\log 2}{\log 3}}(C) \geq \frac{1}{2}.$$

Thus, $\dim_H C \geq \frac{\log 2}{\log 3}$. Indeed, if $\dim_H C$ was $< \frac{\log 2}{\log 3}$, we would have $H_{\frac{\log 2}{\log 3}}(C) = 0$.